

Multiplicity result for the fractional Kirchhoff type equations with critical nonlinearity

Pawan Kumar Mishra¹

Department of Mathematics,
Indian Institute of Technology Delhi
Hauz Khaz, New Delhi-16, India.

Abstract

We study the following convex-concave type problem with sign changing nonlinearity

$$M \left(\int_{\Omega} |(-\Delta)^{\frac{1}{4}} u|^2 dx \right) (-\Delta)^{\frac{1}{2}} u = \lambda f(x) |u|^{q-2} u + |u|^2 u \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^2 \setminus \Omega,$$

where $M(t) = a + \epsilon t$, $a, \epsilon > 0$, $\Omega \subset \mathbb{R}^2$ is open bounded domain with smooth boundary, $1 < q < 2$, $f \in L^{\frac{4}{4-q}}(\Omega)$ is sign changing and λ is a positive parameter. Using the idea of Nehari manifold technique, we prove the existence of atleast two solutions for sufficiently small choice of ϵ .

Keywords: Fractional Laplacian, Kirchhoff type problem, critical exponent.

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1. Introduction

In this article, we study the following problem for existence and multiplicity of solutions

$$M \left(\int_{\Omega} |(-\Delta)^{\frac{1}{4}} u|^2 dx \right) (-\Delta)^{\frac{1}{2}} u = \lambda f(x) |u|^{q-2} u + |u|^2 u, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^2 \setminus \Omega, \quad (1.1)$$

where $M(t) = a + \epsilon t$, $a, \epsilon > 0$, $\Omega \subset \mathbb{R}^2$ is open bounded domain with smooth boundary, $1 < q < 2$, $f \in L^{\frac{4}{4-q}}(\Omega)$ is sign changing and λ is a positive parameter. Here $(-\Delta)^{\frac{1}{2}}$ is the $\frac{1}{2}$ -Laplacian operator defined as

$$-(-\Delta)^{\frac{1}{2}} u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^2} dy \quad \text{for all } x \in \mathbb{R}^2.$$

Recently a lot of attention has been given to the study of elliptic equations involving fractional Laplace operator and non-local operators because of pure mathematical research and its wide range of applications in many branches of Science. In the last decade many authors studied the existence and multiplicity of solutions for nonlocal problems involving the nonlocal operators like fractional powers of Laplacian $(-\Delta)^s$, $s \in (0, 1)$. We cite [6, 8, 14, 15, 16, 23, 24, 29, 5, 3, 27, 28] with no attempts to provide the complete list of references. Non-local operators, naturally arise in continuum mechanics, phase transition phenomena, population dynamics and game theory, see [7] and references therein. Fractional operators are also involved in financial mathematics, where Levy processes with jumps appear in modeling the asset

¹email: pawanmishra31284@gmail.com

prices, see [1]. The fractional Laplacian is the infinitesimal generator of Lévy stochastic processes. This is also of interest in Fourier analysis where it is defined as a pseudo-differential operator. Moreover, these operators arise in a quite natural way in many different physical situations in which one has to consider long range anomalous diffusions and transport in highly heterogeneous medium.

In the last decade, many researchers have studied the Kirchhoff type problems involving fractional Laplacian, see [13, 19, 2, 21, 22] and references therein. In [13], authors have given the motivation for the fractional Kirchhoff type operators by studying the string vibrations. Moreover using the concentration-compactness principle authors have proved the existence result for the critical exponent nonlinearity.

In the case $M \equiv 1$, the multiplicity results for fractional Laplacian with critical exponent has been studied in [29]. In [29], authors have considered the following problem

$$(-\Delta)^s u(x) = \lambda f(x)|u|^{q-2}u + |u|^{2_s^*-2}u \text{ in } \Omega, \quad u = 0 \quad \text{on } \mathbb{R}^n \setminus \Omega,$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2 > q > 1$, f is continuous sign changing weight, $\lambda > 0$ is a parameter and $2_s^* = \frac{2n}{n-2s}$ is the critical Sobolev exponent. Using the harmonic extension technique authors have extended the multiplicity results obtained in [25] to the nonlocal fractional Laplacian.

In the case of $s = 1$, the Kirchhoff type problems have been addressed by many researchers, see [9, 17, 11, 12] and references therein. Recently in [17], authors have shown the multiplicity result for Kirchhoff type problems with the restriction that the coefficient of the linear term is sufficiently small.

In this paper we have established the multiplicity results for the fractional Kirchhoff problem with the similar restriction on the Kirchhoff term. We have adopted the Nehari manifold technique to obtain the multiplicity of solutions by minimizing the energy functional on the non-empty decompositions of Nehari manifold. For the details related to Nehari manifold and fibering map analysis, see [10, 25, 4, 26]. We have considered the problem in dimension 2 for the case $s = \frac{1}{2}$ because for any $s \in (0, 1)$, we found that it is hard to find a minimizing sequence in \mathcal{N}_λ^- which converges strongly in \mathcal{N}_λ^- , a similar result as in Lemma 3.8 below. Moreover to apply harmonic extension technique we need the extension map to be an isometry, which is not the case for $s \neq \frac{1}{2}$. Therefore to study the problem for all $s \in (0, 1)$, the harmonic extension technique is not applicable. To the best of my knowledge, the multiplicity result for fractional Kirchhoff type problems with critical exponent, obtained in this paper, has not been established before.

With this introduction, we state the main result of the paper in the form of following theorem.

Theorem 1.1. *There exists a $\lambda_0 > 0$ such that problem (1.1) has at least two non-negative solutions for $\lambda \in (0, \lambda_0)$.*

2. Variational formulation and functional setting

In order to prove our main result, we look for the solutions of the problem (1.1) in the space $H_0^{\frac{1}{2}}(\Omega)$ defined as follows.

$$H_0^{\frac{1}{2}}(\Omega) = \{u \in L^2(\Omega) : [u] < \infty\}, \text{ where } [u] = \left(\int_{\Omega} |(-\Delta)^{\frac{1}{4}} u|^2 dx \right)^{\frac{1}{2}}.$$

The space $H_0^{\frac{1}{2}}(\Omega)$ is the Hilbert space with the inner product defined as

$\langle u, v \rangle = \int_{\Omega} (-\Delta)^{\frac{1}{4}} u (-\Delta)^{\frac{1}{4}} v dx$. The variational functional associated to the problem (1.1) is given as

$$J_\lambda(u) = \frac{1}{2} \widehat{M} \left(\int_{\Omega} |(-\Delta)^{\frac{1}{4}} u|^2 dx \right) - \frac{\lambda}{q} \int_{\Omega} f(x) |u|^q dx - \frac{1}{4} \int_{\Omega} |u|^4 dx,$$

where $\widehat{M}(t) = \int_0^t M(s)ds$ is the primitive of M .

Definition 2.1. A function $u \in H_0^{\frac{1}{2}}(\Omega)$ is called a weak solution of the problem (1.1) if for all $\phi \in H_0^{\frac{1}{2}}(\Omega)$ the following holds

$$M \left(\int_{\Omega} |(-\Delta)^{\frac{1}{4}} u|^2 dx \right) \int_{\Omega} (-\Delta)^{\frac{1}{4}} u (-\Delta)^{\frac{1}{4}} \phi dx = \lambda \int_{\Omega} f(x) |u|^{q-2} u \phi dx + \int_{\Omega} |u|^2 u \phi dx.$$

Recently a powerful technique is developed by Caffarelli and Silvestre [8] to treat the nonlocal problems involving fractional Laplacian. In this technique, we study an extension problem corresponding to a nonlocal problem so that we can investigate the nonlocal problem via classical variational methods. In this work we use this harmonic extension technique. We first define the harmonic extension of $u \in H_0^{\frac{1}{2}}(\Omega)$.

Definition 2.2. For $u \in H_0^{\frac{1}{2}}(\Omega)$, the harmonic extension $E_{\frac{1}{2}}(u) := w$ is the solution of the following problem

$$\begin{cases} -\Delta w &= 0 & \text{in } \mathcal{C} = \Omega \times (0, \infty), \\ w &= 0 & \text{on } \partial\Omega \times (0, \infty), \\ w &= u & \text{on } \Omega \times \{0\}. \end{cases}$$

Moreover, $(-\Delta)^{\frac{1}{2}} u(z) = -\lim_{y \rightarrow 0^+} \frac{\partial w}{\partial y}(z, y)$.

The extension map $E_{\frac{1}{2}} : H_0^{\frac{1}{2}}(\Omega) \rightarrow H_{0,L}^1(\mathcal{C})$, where $H_{0,L}^1(\mathcal{C})$ is the completion of $C_0^\infty(\Omega)$ under the norm $\|w\| = \left(\int_{\mathcal{C}} |\nabla w|^2 dz dy \right)^{\frac{1}{2}}$, is an isometry. In the subsequent Lemmas we use the following trace embedding result

Lemma 2.1. Let $2 \leq q \leq 4$, then there exists $C_q > 0$ such that for all $v \in H_{0,L}^1(\mathcal{C})$,

$$\left(\int_{\Omega \times \{0\}} |v(z, 0)|^q dx \right)^{\frac{1}{q}} \leq C_q \left(\int_{\mathcal{C}} |\nabla v|^2 dz dy \right)^{\frac{1}{2}}.$$

Moreover the embedding is compact for $q \in [2, 4)$.

As discussed above, the problem (1.1) is equivalent to the study of the following extension problem

$$\begin{cases} \Delta w &= 0, & \text{in } \mathcal{C} = \Omega \times (0, \infty), \\ w &= 0 & \text{on } \partial\Omega \times (0, \infty), \\ M(\|w\|^2) \frac{\partial w}{\partial \nu} &= \lambda f(z) |w|^{q-2} w + |w|^2 w & \text{on } \Omega \times \{0\}, \end{cases} \quad (2.2)$$

where $\frac{\partial w}{\partial \nu} = -\lim_{y \rightarrow 0^+} \frac{\partial w}{\partial y}(z, y)$.

The variational functional $\mathcal{I}_{M,\lambda} : H_{0,L}^1(\mathcal{C}) \rightarrow \mathbb{R}$ associated to the problem (2.2) is defined as

$$\mathcal{I}_{M,\lambda}(w) = \frac{1}{2} \widehat{M}(\|w\|^2) - \frac{\lambda}{q} \int_{\Omega \times \{0\}} f(z) |w(z, 0)|^q dz - \frac{1}{4} \int_{\Omega \times \{0\}} |w(z, 0)|^4 dz. \quad (2.3)$$

Any function $w \in H_{0,L}^1(\mathcal{C})$ is called the weak solution of the problem (2.2) if for all $\phi \in H_{0,L}^1(\mathcal{C})$

$$M(\|w\|^2) \int_{\mathcal{C}} \nabla w \cdot \nabla \phi dz dy = \lambda \int_{\Omega \times \{0\}} f(z) |w(z, 0)|^{q-2} w(z, 0) \phi(z, 0) dz + \int_{\Omega \times \{0\}} |w(z, 0)|^4 \phi(z, 0) dz.$$

It is clear that critical points of $\mathcal{I}_{M,\lambda}$ in $H_{0,L}^1(\mathcal{C})$ corresponds to the critical points of J_λ in $H_0^{\frac{1}{2}}(\Omega)$. Thus if w solves the problem (2.2), then $u = \text{trace}(w) = w(z, 0)$ is the solution of the problem (1.1) and vice-versa. Therefore we look for the solutions w of extended problem (2.2) to get the solutions of the problem (1.1).

3. Nehari manifold and fibering maps

Now we consider the Nehari manifold associated to the problem (2.2) as

$$\mathcal{N}_\lambda = \{w \in H_{0,L}^1(\mathcal{C}) \setminus \{0\} \mid \langle \mathcal{I}'_{M,\lambda}(w), w \rangle = 0\}.$$

Thus $w \in \mathcal{N}_\lambda$ if and only if

$$M(\|w\|^2) \|w\|^2 - \lambda \int_{\Omega \times \{0\}} f(z) |w(z, 0)|^q dz - \int_{\Omega \times \{0\}} |w(z, 0)|^4 dz = 0. \quad (3.4)$$

Now for a fixed $w \in H_{0,L}^1(\mathcal{C})$ we define the fiber map $\Phi_w : \mathbb{R}^+ \rightarrow \mathbb{R}$ as $\Phi_w(t) = \mathcal{I}_{M,\lambda}(tw)$. Thus $tw \in \mathcal{N}_\lambda$ if and only if

$$\Phi'_w(t) = tM(t^2\|w\|^2) \|w\|^2 - \lambda t^{q-1} \int_{\Omega \times \{0\}} f(z) |w(z, 0)|^q dz - t^3 \int_{\Omega \times \{0\}} |w(z, 0)|^4 dz = 0.$$

Also

$$\Phi''_w(1) = a\|w\|^2 + 3\epsilon\|w\|^4 - (q-1)\lambda \int_{\Omega \times \{0\}} f(z) |w(z, 0)|^q dz - 3 \int_{\Omega \times \{0\}} |w(z, 0)|^4 dz. \quad (3.5)$$

We split \mathcal{N}_λ into three parts as

$$\mathcal{N}_\lambda^\pm = \{w \in \mathcal{N}_\lambda \mid \Phi''_w(1) \gtrless 0\} \text{ and } \mathcal{N}_\lambda^0 = \{w \in \mathcal{N}_\lambda \mid \Phi''_w(1) = 0\}.$$

The following lemma shows that the set \mathcal{N}_λ is a manifold.

Lemma 3.1. *There exists $\lambda_1 > 0$ such that $\mathcal{N}_\lambda^0 = \emptyset$, for all $\lambda \in (0, \lambda_1)$.*

Proof. We have following two cases.

Case 1: $w \in \mathcal{N}_\lambda$ and $\int_{\Omega \times \{0\}} f(z) |w(z, 0)|^q dz = 0$.

From (3.4), we have, $a\|w\|^2 + \epsilon\|w\|^4 - \int_{\Omega \times \{0\}} |w(z, 0)|^4 dz = 0$. Now,

$$\begin{aligned} 2a\|w\|^2 + 4\epsilon\|w\|^4 - 4 \int_{\Omega \times \{0\}} |w|^4 dz &= 2a\|w\|^2 + 4\epsilon\|w\|^4 - 4(a\|w\|^2 + \epsilon\|w\|^4) \\ &= -2a\|w\|^2 < 0 \end{aligned}$$

which implies $w \notin \mathcal{N}_\lambda^0(\Omega)$.

Case 2: $w \in \mathcal{N}_\lambda$ and $\int_{\Omega \times \{0\}} f(z)|w(z,0)|^q dz \neq 0$.

Suppose $w \in \mathcal{N}_\lambda^0$. Then from equations (3.4) and (3.5), we have

$$(2-q)a\|w\|^2 + (4-q)\epsilon\|w\|^4 = (4-q) \int_{\Omega \times \{0\}} |w(z,0)|^4 dz, \quad (3.6)$$

$$2a\|w\|^2 = (4-q)\lambda \int_{\Omega \times \{0\}} f(z)|w(z,0)|^q dz. \quad (3.7)$$

Define $E_\lambda : \mathcal{N}_\lambda \rightarrow \mathbb{R}$ as

$$E_\lambda(w) = \frac{2a\|w\|^2}{4-q} - \lambda \int_{\Omega \times \{0\}} f(z)|w(z,0)|^q dz,$$

then from equation (3.7), $E_\lambda(w) = 0$, for all $w \in \mathcal{N}_\lambda^0$. Also,

$$\begin{aligned} E_\lambda(w) &\geq \frac{a\|w\|^2}{4-q} - \lambda \|f\|_{\frac{4}{4-q}} \|w\|^q S^{-q}, \\ &\geq \|w\|^q \left(\left(\frac{2}{4-q} \right) a\|w\|^{(2-q)} - \lambda \|f\|_{\frac{4}{4-q}} S^{-q} \right), \end{aligned}$$

where S is the best constant of the embedding $S \left(\int_{\Omega \times \{0\}} |w(z,0)|^4 dz \right)^{\frac{1}{2}} \leq \int_{\mathcal{C}} |\nabla w|^2 dz dy$. Now, from equation (3.6), we get

$$\|w\| \geq \left(\left(\frac{2-q}{4-q} \right) a S^4 \right)^{\frac{1}{2}}. \quad (3.8)$$

From equation (3.8), there exists $\lambda_1 > 0$ such that for $\lambda \in (0, \lambda_1)$, $E_\lambda(w) > 0$, $\forall w \in \mathcal{N}_\lambda^0$, which is contradiction. \square

Lemma 3.2. $\mathcal{I}_{M,\lambda}$ is coercive and bounded below on \mathcal{N}_λ . Moreover, there exists a constant $C > 0$ such that $\mathcal{I}_{M,\lambda} > -C\lambda^{4/(4-q)}$.

Proof. For $w \in \mathcal{N}_\lambda$, we have

$$\begin{aligned} \mathcal{I}_{M,\lambda}(w) &= \frac{1}{4}a\|w\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{4} \right) \int_{\Omega \times \{0\}} f(z)|w(z,0)|^q dz, \\ &\geq \frac{1}{4}a\|w\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{4} \right) S^{-q} \|f\|_{\frac{4}{4-q}} \|w\|^q. \end{aligned}$$

Define $g(t) = \frac{1}{4}at^{\frac{2}{q}} - \lambda \left(\frac{1}{q} - \frac{1}{4} \right) S^{-q} \|f\|_{\frac{4}{4-q}} t$, then $g(t)$ attains its minimum at

$$\left(\frac{\lambda(4-q)\|f\|_{\frac{4}{4-q}} S^{-q}}{2a} \right)^{\frac{q}{2-q}}. \text{ Hence } \mathcal{I}_{M,\lambda}(w) \geq -C\lambda^{\frac{2}{2-q}} \text{ for some constant } C > 0. \quad \square$$

Define $H^\pm = \left\{ w \in H_{0,L}^1(\mathcal{C}) : \int_{\Omega \times \{0\}} f(z)|w(z,0)|^q dz \gtrless 0 \right\}$.

Lemma 3.3. (i) For every $w \in H^+$, there is a unique $t_{\max} = t_{\max}(w) > 0$ and unique $t^+(w) < t_{\max} < t^-(w)$ such that $t^+w \in \mathcal{N}_\lambda^+$, $t^-w \in \mathcal{N}_\lambda^-$ and $\mathcal{I}_{M,\lambda}(t^+w) = \min_{0 \leq t \leq t^+} \mathcal{I}_{M,\lambda}(tw)$, $\mathcal{I}_{M,\lambda}(t^-w) = \max_{t \geq t_{\max}} \mathcal{I}_{M,\lambda}(tw)$.
(ii) For $w \in H^-$, there exists a unique $t^* > 0$ such that $t^*w \in \mathcal{N}_\lambda^-$.

Proof. Define $\psi_w : \mathbb{R}^+ \rightarrow \mathbb{R}$ as

$$\begin{aligned} \psi_w(t) &= at^{2-q}\|w\|^2 + \epsilon t^{4-q}\|w\|^4 - t^{4-q} \int_{\Omega \times \{0\}} |w(z, 0)|^4 dz. \text{ Then} \\ \psi'_w(t) &= a(2-q)t^{1-q}\|w\|^2 + \epsilon(4-q)t^{3-q}\|w\|^4 - (4-q)t^{3-q} \int_{\Omega \times \{0\}} |w(z, 0)|^4 dz. \end{aligned} \quad (3.9)$$

We also note that ϕ_{tw} and ψ_w satisfies $\phi''_{tw}(1) = t^{-q-1}\psi'_w(t)$. Let $w \in H^+$. Then from equation (3.9), we note that $\psi_w(t) \rightarrow -\infty$ as $t \rightarrow \infty$ for small $\epsilon > 0$. From equation (3.9), it is easy to see that $\lim_{t \rightarrow 0^+} \psi'_w(t) > 0$ and $\lim_{t \rightarrow \infty} \psi'_w(t) < 0$. From equation (3.9), it can be shown that there exists a unique $t_{\max} = t_{\max}(w) > 0$ such that $\psi_w(t)$ is increasing on $(0, t_{\max})$, decreasing on (t_{\max}, ∞) and $\psi'_w(t_{\max}) = 0$ that is

$$a(2-q)t_{\max}^2\|w\|^2 + \epsilon(4-q)t_{\max}^4\|w\|^4 - (4-q)t_{\max}^4 \int_{\Omega \times \{0\}} |w(z, 0)|^4 dz = 0. \quad (3.10)$$

Now from equation (3.10), we get

$$t_{\max} \geq \frac{1}{\|w\|} \left(\frac{a(2-q)S^4}{(4-q)} \right)^{\frac{1}{2}} := T_1. \quad (3.11)$$

Using inequality (3.11), we have

$$\begin{aligned} \psi_w(t_{\max}) &\geq \psi_w(T_1) \geq aT_1^{2-q}\|w\|^2 - T_1^{4-q} \int_{\Omega \times \{0\}} |w(z, 0)|^4 dz \\ &\geq 2\|w\|^q \left(\frac{a}{4-q} \right)^{\frac{4-q}{2}} ((2-q)S^4)^{\frac{2-q}{2}} > 0. \end{aligned}$$

Hence if $\lambda < \lambda_2 = \left(\frac{2S^{\frac{-q}{2}}}{\|f\|_{L^{\frac{4}{4-q}}(\Omega)}} \right) \left(\frac{a}{4-q} \right)^{\frac{4-q}{2}} ((2-q)S^4)^{\frac{2-q}{2}}$, there exists unique $t^+ = t^+(w) < t_{\max}$ and $t^- = t^-(w) > t_{\max}$, such that $\psi_w(t^+) = \lambda \int_{\Omega \times \{0\}} f(z)|w|^q dz = \psi_w(t^-)$. That is, $t^+w, t^-w \in \mathcal{N}_\lambda$. Also $\psi'_w(t^+) > 0$ and $\psi'_w(t^-) < 0$ implies $t^+w \in \mathcal{N}_\lambda^+$ and $t^-w \in \mathcal{N}_\lambda^-$. Since $\Phi'_w(t) = t^q \left(\psi_w(t) - \lambda \int_{\Omega \times \{0\}} f(z)|w|^q dz \right)$. Then $\Phi'_w(t) < 0$ for all $t \in [0, t^+)$ and $\Phi'_w(t) > 0$ for all $t \in (t^+, t^-)$. So $\mathcal{I}_{M,\lambda}f(t^+w) = \min_{0 \leq t \leq t^+} \mathcal{I}_{M,\lambda}(tw)$. Also $\Phi'_w(t) > 0$ for all $t \in [t^+, t^-)$, $\Phi'_w(t^-) = 0$ and $\Phi'_w(t) < 0$ for all $t \in (t^-, \infty)$ implies that $\mathcal{I}_{M,\lambda}(t^-w) = \max_{t \geq t_{\max}} \mathcal{I}_{M,\lambda}(tw)$.

(ii) Let $w \in H^-$. Then from equation (3.9), we note that $\psi_w(t) \rightarrow -\infty$ as $t \rightarrow \infty$. Also $\psi'_w(t) > 0$ for all $t > 0$. Hence for all $\lambda > 0$ there exists $t^* > 0$ such that $t^*w \in \mathcal{N}_\lambda^-$. \square

Define $\theta_\lambda = \inf\{\mathcal{I}_{M,\lambda}(w) : w \in \mathcal{N}_\lambda\}$ and $\theta_\lambda^\pm = \inf\{\mathcal{I}_{M,\lambda}(w) : w \in \mathcal{N}_\lambda^\pm\}$ then we have the following Lemma.

Lemma 3.4. There exists $C > 0$ such that $\theta_\lambda^+ < -\frac{a(2-q)}{4q}C < 0$.

Proof. Let $v_\lambda \in H_{0,L}^1(\mathcal{C})$ such that $\int_{\Omega \times \{0\}} f(z)|v_\lambda|^q dz > 0$. Then by Lemma 3.3, there exists unique $t_\lambda(v_\lambda) > 0$ such that $t_\lambda v_\lambda \in \mathcal{N}_\lambda^+$. Now from equations (3.4) and (3.5), we have

$$\mathcal{I}_{M,\lambda}(t_\lambda v_\lambda) = \left(\frac{1}{2} - \frac{1}{q}\right) a \|t_\lambda v_\lambda\|^2 + \left(\frac{1}{4} - \frac{1}{q}\right) \epsilon \|t_\lambda v_\lambda\|^4 + \left(\frac{1}{q} - \frac{1}{4}\right) \int_{\Omega \times \{0\}} |t_\lambda v_\lambda|^4 dz.$$

and

$$\int_{\Omega \times \{0\}} |t_\lambda v_\lambda|^4 dz \leq \left(\frac{2-q}{4-q}\right) a \|t_\lambda v_\lambda\|^2 + \epsilon \|t_\lambda v_\lambda\|^4.$$

Therefore

$$\mathcal{I}_{M,\lambda}(t_\lambda v_\lambda) \leq -\frac{(2-q)}{4q} a \|t_\lambda v_\lambda\|^2 \leq -\frac{(2-q)}{4q} a C,$$

where $C = \|t_\lambda v_\lambda\|^2$. This implies $\theta_\lambda^+ \leq -\frac{(2-q)}{4q} a C$. \square

Lemma 3.5. *For a given $w \in \mathcal{N}_\lambda$ and $\lambda \in (0, \lambda_0)$, there exists $\epsilon > 0$ and a differentiable function $\xi : \mathcal{B}(0, \epsilon) \subseteq H_{0,L}^1(\mathcal{C}) \rightarrow \mathbb{R}$ such that $\xi(0) = 1$, the function $\xi(v)(u - v) \in \mathcal{N}_\lambda$ and*

$$\langle \xi'(0), v \rangle = \frac{2a \langle w, v \rangle + 4\epsilon \|w\| \langle w, v \rangle - q\lambda \int_{\Omega \times \{0\}} f(z)|w|^{q-2} w v dz - 4 \int_{\Omega \times \{0\}} |w|^2 w v dz}{(2-q)a \|w\|^2 + (4-q)\epsilon \|w\|^4 - (4-q) \int_{\Omega \times \{0\}} |w|^4 dz}, \quad (3.12)$$

where $\langle w, v \rangle = \int_{\mathcal{C}} \nabla w \nabla v dz dy$ for all $v \in H_{0,L}^1(\mathcal{C})$.

Proof. For fixed $u \in \mathcal{N}_\lambda$, define $\mathcal{F}_u : \mathbb{R} \times H_{0,L}^1(\mathcal{C}) \rightarrow \mathbb{R}$ as follows

$$\mathcal{F}_u(t, w) = t^2 a \|u - w\|^2 + t^4 \epsilon \|u - w\|^4 - t^q \lambda \int_{\Omega \times \{0\}} f(z)|u - w|^q dz - t^4 \int_{\Omega \times \{0\}} |u - w|^4 dz,$$

then $\mathcal{F}_u(1, 0) = 0$, $\frac{\partial}{\partial t} \mathcal{F}_u(1, 0) \neq 0$ as $\mathcal{N}_\lambda^0 = \emptyset$. So we can apply implicit function theorem to get a differentiable function $\xi : \mathcal{B}(0, \epsilon) \subseteq H_{0,L}^1(\mathcal{C}) \rightarrow \mathbb{R}$ such that $\xi(0) = 1$ and equation (3.12) holds and $\mathcal{F}_u(\xi(w), w) = 0$, for all $w \in \mathcal{B}(0, \epsilon)$.

Hence $\xi(w)(u - w) \in \mathcal{N}_\lambda$. \square

Now using the Lemma 3.5, we prove the following proposition which shows the existence of Palais-Smale sequence.

Proposition 3.1. *Let $\lambda \in (0, \lambda_1)$. Then there exists a minimizing sequence $\{w_k\} \subset \mathcal{N}_\lambda$ such that*

$$\mathcal{I}_{M,\lambda}(w_k) = \theta_\lambda + o_k(1) \text{ and } \mathcal{I}'_{M,\lambda}(w_k) = o_k(1).$$

Proof. From Lemma 3.2, $\mathcal{I}_{M,\lambda}$ is bounded below on \mathcal{N}_λ . So by Ekeland variational principle, there exists a minimizing sequence $\{w_k\} \in \mathcal{N}_\lambda$ such that

$$\begin{aligned} \mathcal{I}_{M,\lambda}(w_k) &\leq \theta_\lambda + \frac{1}{k}, \\ \mathcal{I}_{M,\lambda}(v) &\geq \mathcal{I}_{M,\lambda}(w_k) - \frac{1}{k} \|v - w_k\| \text{ for all } v \in \mathcal{N}_\lambda. \end{aligned} \quad (3.13)$$

Using equation (3.13) and Lemma 3.4, it is easy to show that $w_k \not\equiv 0$. From Lemma 3.2, we have that $\sup_k \|w_k\| < \infty$. Next we claim that $\|\mathcal{I}'_{M,\lambda}(w_k)\| \rightarrow 0$ as $k \rightarrow 0$. Now, using the Lemma 3.5 we get the differentiable functions $\xi_k : \mathcal{B}(0, \epsilon_k) \rightarrow \mathbb{R}$ for some $\epsilon_k > 0$ such that $\xi_k(v)(w_k - v) \in \mathcal{N}_\lambda$, for all $v \in \mathcal{B}(0, \epsilon_k)$. For fixed k , choose $0 < \rho < \epsilon_k$. Let $w \in H_{0,L}^1(\mathcal{C})$ with $w \not\equiv 0$ and let $v_\rho = \frac{\rho w}{\|w\|}$. We set $\eta_\rho = \xi_k(v_\rho)(w_k - v_\rho)$. Since $\eta_\rho \in \mathcal{N}_\lambda$, we get from equation (3.4)

$$\mathcal{I}_{M,\lambda}(\eta_\rho) - \mathcal{I}_{M,\lambda}(w_k) \geq -\frac{1}{k}\|\eta_\rho - w_k\|.$$

Now by mean value theorem, we get

$$\langle \mathcal{I}'_{M,\lambda}(w_k), \eta_\rho - w_k \rangle + o_k(\|\eta_\rho - w_k\|) \geq -\frac{1}{k}\|\eta_\rho - w_k\|.$$

Hence

$$\langle \mathcal{I}'_{M,\lambda}(w_k), -v_\rho \rangle + (\xi_k(v_\rho) - 1)\langle \mathcal{I}'_{M,\lambda}(w_k), (w_k - v_\rho) \rangle \geq -\frac{1}{k}\|\eta_\rho - w_k\| + o_k(\|\eta_\rho - w_k\|)$$

and since $\langle \mathcal{I}'_{M,\lambda}(\eta_\rho), (w_k - v_\rho) \rangle = 0$, we have

$$\begin{aligned} -\rho \langle \mathcal{I}'_{M,\lambda}(w_k), \frac{w}{\|w\|} \rangle + (\xi_k(v_\rho) - 1)\langle \mathcal{I}'_{M,\lambda}(w_k) - \mathcal{I}'_{M,\lambda}(\eta_\rho), (w_k - v_\rho) \rangle \\ \geq -\frac{1}{k}\|\eta_\rho - w_k\| + o_k(\|\eta_\rho - w_k\|). \end{aligned}$$

Thus

$$\begin{aligned} \langle \mathcal{I}'_{M,\lambda}(w_k), \frac{w}{\|w\|} \rangle &\leq \frac{1}{k\rho}\|\eta_\rho - w_k\| + \frac{o_k(\|\eta_\rho - w_k\|)}{\rho} \\ &\quad + \frac{(\xi_k(v_\rho) - 1)}{\rho}\langle \mathcal{I}'_{M,\lambda}(w_k) - \mathcal{I}'_{M,\lambda}(\eta_\rho), (w_k - v_\rho) \rangle. \end{aligned} \quad (3.14)$$

Since $\|\eta_\rho - w_k\| \leq \rho|\xi_k(v_\rho)| + |\xi_k(v_\rho) - 1|\|w_k\|$ and $\lim_{\rho \rightarrow 0^+} \frac{|\xi_k(v_\rho) - 1|}{\rho} \leq \|\xi'_k(0)\|$, taking limit $\rho \rightarrow 0^+$ in (3.14), we get

$$\langle \mathcal{I}'_{M,\lambda}(w_k), \frac{w}{\|w\|} \rangle \leq \frac{C}{k}(1 + \|\xi'_k(0)\|)$$

for some constant $C > 0$, independent of w . So if we can show that $\|\xi'_k(0)\|$ is bounded then we are done. Now from Lemma 3.5, (Note that from Lemma 3.2 and Lemma 3.4, $\|w_k\| \leq C\lambda$) the boundedness of $\{w_k\}$ and Hölder's inequality, for some $K > 0$, we get for $0 < \lambda < \lambda_3$ with $\lambda_3 < \lambda_1$ small enough

$$\langle \xi'(0), v \rangle = \frac{K\|v\|}{(2-q)a\|w_k\|^2 + (4-q)\epsilon\|w_k\|^4 - (4-q) \int_{\Omega \times \{0\}} |w_k|^4 dz}.$$

So to prove the claim we only need to prove that the denominator in the above expression is bounded away from zero. Suppose not. Then there exists a subsequence, still denoted by $\{w_k\}$, such that

$$(2-q)a\|w_k\|^2 + (4-q)\epsilon\|w_k\|^4 - (4-q) \int_{\Omega \times \{0\}} |w_k|^4 dz = o_k(1). \quad (3.15)$$

From equation (3.15) we get $E_\lambda(w_k) = o_k(1)$. Now using the fact that $\|w_k\| \geq C > 0$ and following the proof of Lemma 3.1 we get $E_\lambda(w_k) > C_1$ for all k for some $C_1 > 0$, which is a contradiction. \square

Now we prove the following proposition which shows the compactness of Palais-Smale sequence.

Lemma 3.6. *Suppose $\{w_k\}$ be a sequence in $H_{0,L}^1(\mathcal{C})$ such that $\mathcal{I}_{M,\lambda}(w_k) \rightarrow c < \frac{1}{4}a^2S^2 - C\lambda^{\frac{2}{2-q}}$ and $\mathcal{I}'_{M,\lambda}(w_k) \rightarrow 0$, where C is a positive constant, then there exists a strongly convergent subsequence.*

Proof. Let $\{w_k\}$ be a $(PS)_c$ sequence for $\mathcal{I}_{M,\lambda}$ in $H_{0,L}^1(\mathcal{C})$ then it is easy to see that $\{w_k\}$ is bounded in $H_{0,L}^1(\mathcal{C})$. Therefore there exists $w_0 \in H_{0,L}^1(\mathcal{C})$ such that $w_k \rightharpoonup w_0$ weakly in $H_{0,L}^1(\mathcal{C})$, $w_k(z, 0) \rightarrow w_0(z, 0)$ in $L^\gamma(\Omega)$ and $w_k \rightarrow w_0$ pointwise almost everywhere in $\Omega \times \{0\}$. As $\{w_k\}$ is bounded in $H_{0,L}^1(\mathcal{C})$, we may assume that there exists two positive measures μ and ν on \mathcal{C} such that

$$|\nabla w_k|^2 dz \rightharpoonup d\mu \quad \text{and} \quad |w_k|^4 \rightharpoonup d\nu, \quad (3.16)$$

Moreover, (see, [20]), we have a countable index set J , positive constants $\{\nu_j\}_{j \in J}$ and $\{\mu_j\}_{j \in J}$ such that

$$\nu = |w_0|^4 dz + \sum_{j \in J} \nu_j \delta_{z_j}, \quad \text{and} \quad \mu \geq |\nabla w_0|^2 dz + \sum_{j \in J} \mu_j \delta_{z_j}, \quad \mu_j \geq S\nu_j^{\frac{1}{2}}.$$

Our goal is to show that J is empty. Suppose not then for any $j \in J$ we can consider the cutoff functions, $\psi_{\epsilon,j}(z)$, centered at z_j such that $0 \leq \psi_{\epsilon,j}(z) \leq 1$, $\psi_{\epsilon,j}(z) = 1$ in $B_{\frac{\epsilon}{2}}(z_j)$, $\psi_{\epsilon,j}(z) = 0$ in $B_\epsilon^c(z_j)$, and $|\nabla \psi_{\epsilon,j}(z)| \leq \frac{4}{\epsilon}$. Then we have

$$\lim_{\epsilon \rightarrow 0} \lim_{k \rightarrow \infty} \int_{\Omega \times \{0\}} f(z) |w_k(z, 0)|^{q-1} \psi_{\epsilon,j} w_k(z, 0) dz = 0. \quad (3.17)$$

Now using (3.17)

$$\begin{aligned} 0 &= \lim_{\epsilon \rightarrow \infty} \lim_{k \rightarrow \infty} \langle \mathcal{I}_{M,\lambda}(w_k), \psi_{\epsilon,j} w_k \rangle \\ &= \lim_{\epsilon \rightarrow \infty} \lim_{k \rightarrow \infty} \left\{ (a + \epsilon \|w_k\|^2) \int_{\mathcal{C}} \nabla w_k \nabla (\psi_{\epsilon,j} w_k) dz dy - \int_{\Omega \times \{0\}} |w_k(z, 0)|^4 \psi_{\epsilon,j} dz \right\} \\ &\geq \lim_{\epsilon \rightarrow \infty} \lim_{k \rightarrow \infty} \left\{ (a + \epsilon \|w_k\|^2) \int_{\mathcal{C}} (|\nabla w_k|^2 \psi_{\epsilon,j} + w_k \nabla \psi_{\epsilon,j} \nabla w_k) dz dy - \int_{\Omega \times \{0\}} |w_k(z, 0)|^4 \psi_{\epsilon,j} dz \right\} \\ &\geq a\mu_j - \nu_j. \end{aligned}$$

From the relation $\mu_j \geq S\nu_j^{\frac{1}{2}}$, $\mu_j \geq aS^2$ or $\mu_j = 0$. We claim that $\mu_j \geq aS^2$ is not possible to hold.

$$\begin{aligned} c &= \lim_{k \rightarrow \infty} \left\{ \mathcal{I}_{M,\lambda}(w_k) - \frac{1}{4} \langle \mathcal{I}'_{M,\lambda}(w_k), w_k \rangle \right\} \\ &\geq \frac{1}{4} a \left(\|w_0\|^2 + \sum_{j \in J} \mu_j \right) - \lambda \left(\frac{1}{q} - \frac{1}{4} \right) \int_{\Omega \times \{0\}} f(z) |w_0(z, 0)|^q dz \\ &\geq a\mu_{j_0} + \frac{1}{4} a \|w_0\|^2 - \lambda \left(\frac{1}{q} - \frac{1}{4} \right) \|f\|_{L^{\frac{4}{4-q}}} S^{\frac{-q}{2}} \|w_0\|^q \geq \frac{1}{4} a^2 S^2 - D\lambda^{\frac{2}{2-q}}, \end{aligned}$$

where $D = \left(\frac{(4-q)\|f\|_{\frac{4}{3-q}} S^{\frac{-q}{2}}}{4q} \right)^{\frac{2}{2-q}} \left(\frac{2-q}{2} \right) \left(\frac{2q}{a} \right)^{\frac{q}{2-q}}$ which is a contradiction as $c < \frac{1}{4}a^2S^2 - D\lambda^{\frac{2}{2-q}}$. Hence J is empty and $\int_{\Omega \times \{0\}} |w_k(z, 0)|^4 dz \rightarrow \int_{\Omega \times \{0\}} |w_0(z, 0)|^4 dz$. \square

Existence of first solution: Assume $\lambda_0 = \min\{\lambda_1, \lambda_2, \lambda_3\}$. Now as the functional is bounded below in \mathcal{N}_λ , we minimize the functional in \mathcal{N}_λ^+ and using the Lemma 3.4 and Lemma 3.6 we get the first solution w_0 in \mathcal{N}_λ^+ for $\lambda \in (0, \lambda_0)$. Now the following Lemma shows that w_0 is indeed a local minimizer of $\mathcal{I}_{M,\lambda}$ in $H_{0,L}^1(\mathcal{C})$.

Lemma 3.7. *The function $w_0 \in \mathcal{N}_\lambda^+$ is a local minimum of $\mathcal{I}_{M,\lambda}(w)$ in $H_{0,L}^1(\mathcal{C})$ for $\lambda < \lambda_0$. Moreover $w_0 \geq 0$.*

Proof. Since $w_0 \in \mathcal{N}_\lambda^+$, we have $t^+(w_0) = 1 < t_*(w_0)$. Hence by continuity of $w \mapsto t_*(w)$, given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $1 + \epsilon < t_*(w_0 - w)$ for all $\|w\| < \delta$. Also, from Lemma 3.5 we have, for $\delta > 0$ small enough, we obtain a C^1 map $t : \mathcal{B}(0, \delta) \rightarrow \mathbb{R}^+$ such that $t(w)(w_0 - w) \in \mathcal{N}_\lambda$, $t(0) = 1$. Therefore, for $\delta > 0$ small enough we have $t^+(w_0 - w) = t(w) < 1 + \epsilon < t_*(w_0 - w)$ for all $\|w\| < \delta$. Since $t_*(w_0 - w) > 1$, we obtain $\mathcal{I}_{M,\lambda}(w_0) \leq \mathcal{I}_{M,\lambda}(t_1(w_0 - w)(w_0 - w)) \leq \mathcal{I}_{M,\lambda}(w_0 - w)$ for all $\|w\| < \delta$. This shows that w_0 is a local minimizer for $\mathcal{I}_{M,\lambda}$ in $H_{0,L}^1(\mathcal{C})$. Moreover $\mathcal{I}_{M,\lambda}(w) = \mathcal{I}_{M,\lambda}(|w|)$. Therefore without loss of generality we can assume that w_0 is non-negative local minimizer of $\mathcal{I}_{M,\lambda}$ in $H_{0,L}^1(\mathcal{C})$. \square

Now we show the existence of second solution in \mathcal{N}_λ^- . The following Lemma gives the critical level to show the second solution by considering the mountain pass structure from first solution.

Now consider extremal functions for the fractional Sobolev embedding in \mathbb{R}^2 as $u_\epsilon(z) = \frac{\sqrt{\epsilon}}{\sqrt{\epsilon^2 + |z|^2}}$, $\epsilon > 0$ and $z \in \mathbb{R}^2$. Let $\Sigma = \{z \in \Omega \mid f(z) > 0\}$ be an open set with positive measure. Assume Σ is a domain. Consider the test functions as $\eta \in C_c^\infty(\mathcal{C}_\Sigma)$, where $\mathcal{C}_\Sigma = \Sigma \times (0, \infty)$ such that $0 \leq \eta(z, y) \leq 1$ in \mathcal{C}_Σ and $(\text{supp } f^+ \times \{y > 0\}) \cap \{(z, y) \in \mathcal{C}_\Sigma : \eta = 1\} \neq \emptyset$. Moreover for $\rho > 0$ small, $\eta(z, y) = 1$ on $\mathcal{B}_\rho(0)$ and $\eta(z, y) = 0$ on $\mathcal{B}_{2\rho}^c(0)$. We take ρ small enough such that $\mathcal{B}_{2\rho}(0) \subset \mathcal{C}_\Sigma$. Consider $w_{\epsilon,\eta} = \eta w_\epsilon \in H_{0,L}^1(\mathcal{C})$ then for $\lambda \in (0, \lambda_0)$, we have the following lemma.

Lemma 3.8. *Let w_0 be the local minimum for the functional $\mathcal{I}_{M,\lambda}$ in $H_{0,L}^1(\mathcal{C})$. Then for every $r > 0$ and a.e. $\eta \in \Sigma$ there exists $\epsilon_0 = \epsilon_0(r, \eta) > 0$ s.t.*

$$\mathcal{I}_{M,\lambda}(w_0 + r w_{\epsilon,\eta}) < \frac{1}{4}a^2S^2 - D\lambda^{\frac{1}{2-q}} \text{ for all } \epsilon \in (0, \epsilon_0).$$

Proof. From equation (2.3)

$$\begin{aligned} \mathcal{I}_{M,\lambda}(w_0 + r w_{\epsilon,\eta}) &= \frac{a}{2}\|w_0 + r w_{\epsilon,\eta}\|^2 + \frac{\epsilon}{4}\|w_0 + r w_{\epsilon,\eta}\|^4 - \frac{\lambda}{q} \int_{\Omega \times \{0\}} f(z)|w_0 + r w_{\epsilon,\eta}|^q dz \\ &\quad - \frac{1}{4} \int_{\Omega \times \{0\}} |w_0 + r w_{\epsilon,\eta}|^4 dz \\ &= \frac{a}{2}\|w_0\|^2 + \frac{a}{2}r^2\|w_{\epsilon,\eta}\|^2 + a r \langle w_0, w_{\epsilon,\eta} \rangle + \frac{\epsilon}{4}\|w_0\|^4 + \frac{\epsilon}{4}r^4\|w_{\epsilon,\eta}\|^4 \\ &\quad + \epsilon r^2 \langle w_0, w_{\epsilon,\eta} \rangle^2 + \frac{\epsilon}{2}r^2\|w_0\|^2\|w_{\epsilon,\eta}\|^2 + \epsilon r^3\|w_{\epsilon,\eta}\|^2 \langle w_0, w_{\epsilon,\eta} \rangle \\ &\quad + \epsilon r\|w_0\|^2 \langle w_0, w_{\epsilon,\eta} \rangle - \frac{\lambda}{q} \int_{\Omega \times \{0\}} f(z)|w_0 + r w_{\epsilon,\eta}|^q dz - \frac{1}{4} \int_{\Omega \times \{0\}} |w_0 + r w_{\epsilon,\eta}|^4 dz. \end{aligned}$$

Using the fact that w_0 is a solution of problem (2.2), we obtain

$$\begin{aligned} \mathcal{I}_{M,\lambda}(w_0 + r w_{\epsilon,\eta}) &\leq \mathcal{I}_{M,\lambda}(w_0) + \frac{a}{2}r^2\|w_{\epsilon,\eta}\|^2 + \frac{\epsilon}{4}r^4\|w_{\epsilon,\eta}\|^4 + \epsilon r^2\|w_0\|^2\|w_{\epsilon,\eta}\|^2 + \frac{\epsilon}{2}r^2\|w_0\|^2\|w_{\epsilon,\eta}\|^2 \\ &\quad + \epsilon r^3\|w_{\epsilon,\eta}\|^3\|w_0\| - \frac{\lambda}{q} \left(\int_{\Omega \times \{0\}} f(z)(|w_0 + r w_{\epsilon,\eta}|^q - |w_0|^q - qr|w_0|^{q-1}w_{\epsilon,\eta})dz \right) \\ &\quad - \frac{1}{4} \left(\int_{\Omega \times \{0\}} (|w_0 + r w_{\epsilon,\eta}|^4 - |w_0|^4 - 4rw_0^3w_{\epsilon,\eta})dz \right) \leq \mathcal{I}_{M,\lambda}(w_0). \end{aligned}$$

Using f to be positive \sum and $\|w_0\| = R$ together with Young's inequality, we get

$$\begin{aligned} \mathcal{I}_{M,\lambda}(w_0 + r w_{\epsilon,\eta}) &\leq \mathcal{I}_{M,\lambda}(w_0) + \frac{a}{2}r^2\|w_{\epsilon,\eta}\|^2 + \frac{\epsilon}{4}r^4\|w_{\epsilon,\eta}\|^4 + \frac{3\epsilon}{2}r^2\|w_0\|^2\|w_{\epsilon,\eta}\|^2 \\ &\quad + \epsilon r^3\|w_{\epsilon,\eta}\|^3\|w_0\| - \frac{1}{4} \left(\int_{\Omega \times \{0\}} (|w_0 + r w_{\epsilon,\eta}|^4 - |w_0|^4 - 4rw_0^3w_{\epsilon,\eta})dz \right). \\ &\leq \mathcal{I}_{M,\lambda}(w_0) + \frac{a}{2}r^2\|w_{\epsilon,\eta}\|^2 + \frac{\epsilon}{4}r^4\|w_{\epsilon,\eta}\|^4 + \frac{3\epsilon}{2}r^2\|w_0\|^2\|w_{\epsilon,\eta}\|^2 + \epsilon r^3\|w_{\epsilon,\eta}\|^3\|w_0\| \\ &\quad - \frac{1}{4}r^4 \int_{\Omega \times \{0\}} |w_{\epsilon,\eta}|^4 dz - \frac{2}{3}r^3 \int_{\Omega \times \{0\}} w_0 |w_{\epsilon,\eta}|^3 dz \\ &\leq \mathcal{I}_{M,\lambda}(w_0) + \frac{a}{2}r^2\|w_{\epsilon,\eta}\|^2 + \frac{\epsilon}{4}r^4\|w_{\epsilon,\eta}\|^4 + \frac{3\epsilon}{2}r^2R^2\|w_{\epsilon,\eta}\|^2 + \epsilon R r^3\|w_{\epsilon,\eta}\|^3 \\ &\quad - \frac{1}{4}r^4 \int_{\Omega \times \{0\}} |w_{\epsilon,\eta}|^4 dz - Cr^3 \int_{\Omega \times \{0\}} |w_{\epsilon,\eta}|^3 dz. \\ &\leq \mathcal{I}_{M,\lambda}(w_0) + \frac{a}{2}r^2\|w_{\epsilon,\eta}\|^2 + \frac{\epsilon}{4}r^4\|w_{\epsilon,\eta}\|^4 + \frac{\epsilon}{4}r^4\|w_{\epsilon,\eta}\|^4 + \frac{9\epsilon}{4}R^4 + \frac{3}{4}\epsilon r^4\|w_{\epsilon,\eta}\|^4 + \frac{\epsilon}{4}R^4 \\ &\quad - \frac{1}{4}r^4 \int_{\Omega \times \{0\}} |w_{\epsilon,\eta}|^4 dz - Cr^3 \int_{\Omega \times \{0\}} |w_{\epsilon,\eta}|^3 dz \\ &= \mathcal{I}_{M,\lambda}(w_0) + \frac{a}{2}r^2\|w_{\epsilon,\eta}\|^2 + \frac{5\epsilon}{4}r^4\|w_{\epsilon,\eta}\|^4 - \frac{1}{4}r^4 \int_{\Omega \times \{0\}} |w_{\epsilon,\eta}|^4 dz \\ &\quad - Cr^3 \int_{\Omega \times \{0\}} |w_{\epsilon,\eta}|^3 dz + \frac{5\epsilon}{2}R^4. \end{aligned}$$

Now assume $g(t) = \frac{a}{2}t^2\|w_{\epsilon,\eta}\|^2 + \frac{5\epsilon}{4}t^4\|w_{\epsilon,\eta}\|^4 - \frac{1}{4}t^4 \int_{\Omega \times \{0\}} |w_{\epsilon,\eta}|^4 dz - Ct^3 \int_{\Omega \times \{0\}} |w_{\epsilon,\eta}|^3 dz$.

Since $\lim_{\epsilon \rightarrow 0} \lim_{t \rightarrow \infty} g(t) = -\infty$. Therefore there exists $t_\epsilon > 0$ such that

$$g(t_\epsilon w_{\epsilon,\eta}) = \sup_{t \geq 0} g(t w_{\epsilon,\eta}) \text{ and } \frac{d}{dt} g(t w_{\epsilon,\eta})|_{t=t_\epsilon} = 0. \quad (3.18)$$

From equation (3.18), it is easy to show that there exist constants $t_1, t_2 > 0$ such that $0 < t_1 \leq t_\epsilon \leq t_2 < \infty$

for all $\epsilon > 0$ small enough. Now using the estimate on $\int_{\Omega \times \{0\}} |w_{\epsilon,\eta}|^3 dz = C_1 \epsilon^{\frac{1}{2}} + O(\epsilon^{\frac{3}{2}})$,

$$\begin{aligned} \sup_{t \geq 0} g(t w_{\epsilon,\eta}) &\leq \sup_{t \geq 0} \left(\frac{a}{2}t^2\|w_{\epsilon,\eta}\|^2 - \frac{1}{4}t^4 \int_{\Omega \times \{0\}} |w_{\epsilon,\eta}|^4 dz \right) + \frac{5\epsilon}{4}t_1^4\|w_{\epsilon,\eta}\|^4 - Ct_1^3 \int_{\Omega \times \{0\}} |w_{\epsilon,\eta}|^3 dz \\ &\leq \frac{1}{4}a^2S^2 + C_3\epsilon - C_4\epsilon^{\frac{1}{2}}. \end{aligned}$$

Now using $\mathcal{I}_{M,\lambda}(w_0) < 0$, we get $\mathcal{I}_{M,\lambda}(w_0 + r w_{\epsilon,\eta}) \leq \frac{1}{4}a^2S^2 + C_5\epsilon - C_4\epsilon^{\frac{1}{2}}$. Now choosing $\epsilon = \lambda^{\frac{2}{2-q}}$ and $0 < \lambda < (\frac{C_4}{C_5+D})^{2-q}$, we get $\mathcal{I}_{M,\lambda}(w_0 + r w_{\epsilon,\eta}) \leq \frac{1}{4}a^2S^2 - D\lambda^{\frac{2}{2-q}}$. \square

Now consider the following

$$\begin{aligned} W_1 &= \left\{ w \in H_{0,L}^1(\mathcal{C}) \setminus \{0\} \mid \frac{1}{\|w\|} t^- \left(\frac{w}{\|w\|} \right) > 1 \right\} \cup \{0\}, \\ W_2 &= \left\{ w \in H_{0,L}^1(\mathcal{C}) \setminus \{0\} \mid \frac{1}{\|w\|} t^- \left(\frac{w}{\|w\|} \right) < 1 \right\}. \end{aligned}$$

Then \mathcal{N}_λ^- disconnects $H_{0,L}^1(\mathcal{C})$ in two connected components W_1 and W_2 and $H_{0,L}^1(\mathcal{C}) \setminus \mathcal{N}_\lambda^- = W_1 \cup W_2$. For each $w \in \mathcal{N}_\lambda^+$, we have $1 < t_{\max}(w) < t^-(w)$. Since $t^-(w) = \frac{1}{\|w\|} t^- \left(\frac{w}{\|w\|} \right)$, then $\mathcal{N}_\lambda^+ \subset W_1$. In particular, $w \in W_1$. It can be shown that there exists $l_0 > 0$ such that $w_0^+ + l_0 w_{\epsilon,\eta} \in W_2$. Now, we define $\beta = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} \mathcal{I}_{M,\lambda}(\gamma(s))$, where $\Gamma = \{\gamma \in C([0,1]; H_{0,L}^1(\mathcal{C})) \mid \gamma(0) = w \text{ and } \gamma(1) = w + l_0 w_{\epsilon,\eta}\}$. Define a path $\gamma_0 = w + t l_0 w_{\epsilon,\eta}$ for $t \in [0,1]$, then $\gamma_0 \in \Gamma$ and there exists $t_0 \in (0,1)$ such that $\gamma_0(t_0) \in \mathcal{N}_\lambda^-$, we have $\beta \geq \theta_\lambda^-$. Moreover, by Lemma 3.8, $\theta_\lambda^- \leq \beta < \frac{1}{4}a^2S^2 - D\lambda^{\frac{1}{2-q}}$. Now similar to the Proposition 3.1, one can show the existence of Palais-Smale sequence $\{w_k\} \subset \mathcal{N}_\lambda^-$. Since $\theta_\lambda^- < \frac{1}{4}a^2S^2 - D\lambda^{\frac{1}{2-q}}$ by Proposition 3.6, there exist a subsequence $\{w_k\}$ and $w_1 \in H_{0,L}^1(\mathcal{C})$ such that $w_k \rightarrow w_1$ strongly in $H_{0,L}^1(\mathcal{C})$. Since \mathcal{N}_λ^- is closed, $w_1 \in \mathcal{N}_\lambda^-$ and $\mathcal{I}_{M,\lambda}(w_k) \rightarrow \mathcal{I}_{M,\lambda}(w_1) = \theta_\lambda^-$ as $k \rightarrow \infty$. Therefore w_1 is also a solution. Since $\mathcal{I}_{M,\lambda}(w) = \mathcal{I}_{M,\lambda}(|w|)$, we may assume that w_1 is nonnegative (nontrivial) solution of the problem (2.2).

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